7. BROSS H., Zur Theorie der bewegter Verzetzungen. Phys. Stat. Solidi. 5, 2, 1964.
8. NATSIK V.D. and CHISHKO K.A., Sonic irradiation of dislocations moving on the surface of a crystal. Fiz. Tverd. Tela, 20, 2, 1978.
9. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Series, Sums and Products. Nauka, Moscow, 1963.
10. JAHNKE E., EMDE F. and LÖSCH F., Tafeln Höherer Fuctionen. Teubner, Stuttgart, 1966.
11. LANDAU L.D. and LIFSHITZ E.M., Theoretical Physics. 7, Theory of Elasticity. Nauka, Moscow, 1965.
12. KOSEVICH A.M., Dislocations in the Theory of Elasticity. Nauk. Dumka, Kiev, 1978.

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A LINEAR THEORY OF DOUBLE-LAYER RESIN-METAL SHELLS*

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An asymptotic method is used to derive two-dimensional equations of double-layer shells of arbitrary form. The problem is split into two, simpler problems. A solution for a weak layer of slightly compressible elastic material, such as an elastomer, is obtained in general form and the solution for a two-layer shell reduces, as a result, to solving the problem of a stiff layer under a load which depends on the stress-strain state (SDS) of the weak layer. It is shown that in the case of a weak layer the laws of variation of the quantities required across the thickness may deviate significantly, depending on the dynamic properties of the load, from the laws accepted in the classical theory of shells.

The papers dealing with the problem in question concern themselves, as a rule, with the analysis of the equations of state /1, 2/, or with the study of SDS under kinematic-type conditions on the face surfaces of the shell, making certain assumptions /3/.

1. We shall assume, to fix our ideas, that the outer layer of the shell, of thickness $2h_1$, is composed of an incompressible elastic elastomer (we shall call it the soft layer), and an inner, metal layer of thickness $2h_2$ (we shall call it the stiff layer). The face surfaces of the two-layer shell are subjected to an arbitrary, static or dynamic load.

We will write the initial conditions for the elastomer layer in three-orthogonal coordinates $\alpha_1, \alpha_2, \alpha_3$ where α_1, α_2 are the lines of curvature of the middle surface of the layer and α_3 is a line orthogonal to them

$$\sigma_i^{(1)} = 2\mu e_i^{(1)} + p, \quad \sigma_{ij}^{(1)} = \mu\,(m_i^{(1)} + m_j^{(1)}) \tag{1.1}$$
$$\sigma_3^{(1)} = 2\mu \frac{\partial v_3^{(1)}}{\partial \alpha_3^{(1)}} + p, \quad \sigma_{i3}^{(1)} = \mu\left(\frac{\partial v_i^{(1)}}{\partial \alpha_3^{(1)}} + g_i^{(1)}\right)$$
$$e_1^{(1)} + e_2^{(1)} + \frac{\partial v_3^{(1)}}{\partial \alpha_3^{(1)}} = 0 \tag{1.2}$$
$$L_i^{(1)} + \frac{\partial}{\partial \alpha_3^{(1)}} \sigma_{i3}^{(1)} + \rho_1\omega^2 v_i^{(1)} = 0 \quad -L^{(1)} + F^{(1)} + \frac{\partial \sigma_3^{(1)}}{\partial \alpha_3^{(1)}} + \rho_1\omega^2 v_3^{(1)} = 0 \tag{1.3}$$

Here (1.1) is the equation of state, (1.2) is the condition of incompressibility, (1.3) are the equations of motion, and

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$$L_i^{(1)} = \frac{1}{A_i^{(1)}}\frac{\partial \sigma_i^{(1)}}{\partial \alpha_i^{(1)}} + \frac{1}{A_j^{(1)}}\frac{\partial \sigma_{ij}^{(1)}}{\partial \alpha_j^{(1)}} + k_j^{(1)}(\sigma_i^{(1)} - \sigma_j^{(1)}) + k_i^{(1)}(\sigma_{ij}^{(1)} + \sigma_{ji}^{(1)}) \tag{1.4}$$

$$L^{(1)} = \frac{\sigma_1^{(1)}}{R_1} + \frac{\sigma_2^{(1)}}{R_2}$$

$$F^{(1)} = \frac{1}{A_1^{(1)}}\frac{\partial \sigma_{13}^{(1)}}{\partial \alpha_1^{(1)}} + \frac{1}{A_2^{(1)}}\frac{\partial \sigma_{23}^{(1)}}{\partial \alpha_2^{(1)}} + k_2^{(1)}\sigma_{13}^{(1)} + k_1^{(1)}\sigma_{23}^{(1)}$$

$$e_i^{(1)} = \left(\frac{1}{A_i^{(1)}}\frac{\partial v_i^{(1)}}{\partial \alpha_i^{(1)}} + k_i^{(1)} v_j^{(1)} + \frac{v_3^{(1)}}{R_i^{(1)}}\right)\frac{1}{a_i^{(1)}}$$

$$m_i^{(1)} = \left(\frac{1}{A_j^{(1)}}\frac{\partial v_i^{(1)}}{\partial \alpha_j^{(1)}} - k_j^{(1)} v_j^{(1)}\right)\frac{1}{a_j^{(1)}}$$

$$g_i^{(1)} = \left(\frac{1}{A_i^{(1)}}\frac{\partial v_3^{(1)}}{\partial \alpha_i^{(1)}} - \frac{v_i^{(1)}}{R_i^{(1)}}\right)\frac{1}{a_i^{(1)}}, \quad k_i^{(1)} = \frac{1}{A_i^{(1)}A_j^{(1)}}\frac{\partial A_i^{(1)}}{\partial \alpha_j^{(1)}}$$

$$a_i = 1 + \frac{\alpha_3^{(1)}}{R_i^{(1)}}, \quad i \neq j = 1, 2$$

In formulas (1.1)-(1.4) σ_i, σ_{ij}, σ_{i3}, σ_3 are the components of the symmetric stress tensor, v_i, v_3 are the components of the displacement vector, A_i are the coefficients of the first quadratic form of the middle surface, R_i are the radii of curvature of the coordinate lines α_i and μ is the Lamé coefficient. The superscripts within the parenthesis indicate that the parameter in question refers to the first (elastomer) or second (metal) layer. Every equation with indices i, j combines within it two equations, the first of which is obtained by putting $i = 1$, $j = 2$, and the second by putting $i = 2$, $j = 1$ in it.

Let us write the equations of state for the metal layer

$$\sigma_i^{(2)} = \frac{E}{1-\nu}(e_i^{(2)} + \nu e_j^{(2)}) + \frac{\nu}{1-\nu}\sigma_3^{(2)}, \quad \sigma_{ij}^{(2)} = \frac{E}{2(1+\nu)}(m_i + m_j) \tag{1.5}$$

$$E\frac{\partial v_3^{(2)}}{\partial \alpha_3^{(2)}} = \sigma_3^{(2)} - \nu(\sigma_1^{(2)} + \sigma_2^{(2)}), \quad E\left(\frac{\partial v_i^{(2)}}{\partial \alpha_3^{(2)}} + g_i^{(2)}\right) = 2(1+\nu)\sigma_{i3}^{(2)}$$

The remaining relations for the metal layer are identical with relations (1.3)-(1.4) in which the superscript (1) should be replaced by (2).

We specify the components of the surface load on the upper face surface Γ_1 and lower face surface Γ_2 of a two-layer shell as follows:

$$\sigma_3^{(1)}|_{\Gamma_1} = q_3^+, \quad \sigma_{i3}^{(1)}|_{\Gamma_1} = q_{i3}^+ \tag{1.6}$$

$$\sigma_3^{(2)}|_{\Gamma_2} = -q_3^-, \quad \sigma_{i3}^{(2)}|_{\Gamma_2} = -q_{i3}^- \tag{1.7}$$

and the following conditions must hold on the contact surface Γ between the two layers:

$$v_i^{(1)}|_\Gamma = v_i^{(2)}|_\Gamma, \quad v_3^{(1)}|_\Gamma = v_3^{(2)}|_\Gamma, \quad \sigma_3^{(1)}|_\Gamma = \sigma_3^{(2)}|_\Gamma, \quad \sigma_{i3}^{(1)}|_\Gamma = \sigma_{i3}^{(2)}|_\Gamma \tag{1.8}$$

2. Let us consider a static system. We introduce the notation

$$2\mu/E = \eta^a, \; \rho_1/\rho_2 = \eta^b \tag{2.1}$$

Here η is a small parameter equal to the ratio of the half-thickness of the shell to its characteristic dimension R. We assume that the thicknesses of both layers are of the same order of magnitude.

Let us stretch the scale asymptotically along the coordinate lines

$$\alpha_i^{(k)} = R\eta^t \xi_i^{(k)}, \; \alpha_3^{(k)} = R\eta^d \zeta^{(k)} \quad (i, k = 1, 2) \tag{2.2}$$

Here t, which is the index of variability along the coordinates α_i, α_3, is chosen so that the differentiation with respect to ξ_i, ζ does not lead to any significant increases or decreases in the values of the functions required.

We shall assume that the quantity d is equal to unity, and we shall find it of use in what follows.

The following asymptotic representation of the quantities required leads to a non-contradictory approximate solution of the problem. This enables us to determine the SDS of each layer, taking the boundary conditions into account, and enables us to set up an iteration process in order to increase the accuracy of the solution obtained in the initial approximation for the elastomer layer

$$\frac{\sigma_i^{(1)}}{E} = \eta^{-c}\sigma_{i*}^{(1)}, \quad \frac{p}{E} = \eta^{-c}p_*, \quad \frac{\sigma_{ij}^{(1)}}{E} = \eta^{d-2t+a}\sigma_{ij*}^{(1)} \tag{2.3}$$
$$\frac{\sigma_{i3}^{(1)}}{E} = \eta^{-t}\sigma_{i3*}^{(1)}, \quad \frac{\sigma_3^{(1)}}{E} = \eta^{-c}\sigma_{3*}^{(1)}, \quad \frac{v_i^{(1)}}{R} = \eta^{d-t-a}v_{i*}^{(1)}, \quad \frac{v_3^{(1)}}{R} = \eta^{e-a}v_{3*}^{(1)}$$

and the metal layer /4/

$$\frac{\sigma_i^{(2)}}{E} = \eta^{-1}\sigma_{i*}^{(2)}, \quad \frac{\sigma_{ij}^{(2)}}{E} = \eta^{-1}\sigma_{ij*}^{(2)}, \quad \frac{v_i^{(2)}}{R} = \eta^{-1+t}v_{i*}^{(2)} \tag{2.4}$$
$$\frac{v_3^{(2)}}{R} = \eta^{c-1}v_{3*}^{(2)}, \quad \frac{\sigma_{i3}^{(2)}}{E} = \eta^{-t}\sigma_{i3*}^{(2)}, \quad \frac{\sigma_3^{(2)}}{E} = \eta^{-c}\sigma_{3*}^{(2)}$$
$$c = \begin{cases} 0, & t < 1/2 \\ -d + 2t, & t \geqslant 1/2 \end{cases}, \quad e = \begin{cases} 2d - 2t, & a - 2d + 2t - 1 + c \geqslant 0 \\ a - 1 + c, & a - 2d + 2t - 1 + c < 0 \end{cases}$$
$$d - c \leqslant e \leqslant 2d - 2t$$

The powers of η accompanying the dimensionless stresses and displacements required are chosen so as to make the quantities marked with an asterisk of the same order of magnitude.

Formulas (2.3) and (2.4) correspond to the SDS generated by the surface load which has the following asymptotic representation:

$$\pm q_3{\pm} = \eta^{-t}(\pm q_{3*}^{\pm}E), \quad \pm q_{i3}{\pm} = \eta^{-c}(\pm q_{i3*}^{\pm}E) \tag{2.5}$$

Here we assume that to every separate component of the surface load there corresponds an SDS with the asymptotic representations (2.3), (2.4), i.e. a tangential surface load $O(\eta^{-c})$ creates an SDS of the same order of the normal loading $O(\eta^{-t})$.

Let us rewrite, taking into account the asymptotic formulas (2.3)-(2.5), the conditions on the face surfaces of the shell and at the contact surface between the layers:

$$\sigma_{i3*}^{(1)}|_{\Gamma_1} = q_{i3*}^{+}, \quad \sigma_{3*}^{(1)}|_{\Gamma_1} = q_{3*}^{+}, \quad \sigma_{i3*}^{(2)}|_{\Gamma_2} = -q_{i3*}^{-}, \quad \sigma_{3*}^{(2)}|_{\Gamma_2} = -q_{3*}^{-} \tag{2.6}$$
$$v_{i*}^{(1)}|_\Gamma = \eta^{a-d-1+2t}v_{i*}^{(2)}|_\Gamma, \quad v_{3*}^{(1)}|_\Gamma = \eta^{a+c-1-e}v_{3*}^{(2)}|_\Gamma$$
$$\sigma_{i3*}^{(1)}|_\Gamma = \sigma_{i3*}^{(2)}|_\Gamma, \quad \sigma_{3*}^{(1)}|_\Gamma = \sigma_{3*}^{(2)}|_\Gamma$$

Taking into account the asymptotic formula (2.3) and the change of variables (2.2), we obtain Eqs.(1.1)-(1.4) in the form

$$\sigma_{i*}^{(1)} = \eta^{d-2t+c}e_{i*}^{(1)} + p_*, \quad \sigma_{ij*}^{(1)} = \frac{1}{2}(m_{i*}^{(1)} + m_{j*}^{(1)}) \tag{2.7}$$
$$\eta^{e-d+c}\frac{\partial v_{3*}^{(1)}}{\partial\zeta^{(1)}} = \sigma_{3*}^{(1)} - p_*^{(1)}, \quad \frac{\partial v_{i*}^{(1)}}{\partial\zeta^{(1)}} = -\eta^{-d-c}g_{i*}^{(1)} + 2\sigma_{i3*}^{(1)}$$
$$\eta^{2d-2t-e}(e_{1*}^{(1)} + e_{2*}^{(1)}) + \frac{\partial v_{3*}^{(1)}}{\partial\zeta} = 0$$
$$\eta^{d-c}L_{i*}^{(1)} + \frac{\partial\sigma_{i3*}^{(1)}}{\partial\zeta^{(1)}} = 0, \quad -\eta^d L_*^{(1)} + \eta^{d-2t+c}F_*^{(1)} + \frac{\partial\sigma_{3*}^{(1)}}{\partial\zeta^{(1)}} = 0$$

where

$$e_{i*}^{(1)} = \frac{1}{a_{i*}^{(1)}}\left(\frac{1}{A_i^{(1)}}\frac{\partial v_{i*}^{(1)}}{\partial\xi_i^{(1)}} + \eta^t Rk_i^{(1)}v_{j*}^{(1)} + \eta^{e-d+2t}R\frac{v_{3*}^{(1)}}{R_i^{(1)}}\right) \tag{2.8}$$
$$m_{i*}^{(1)} = \frac{1}{a_{j*}}\left(\frac{1}{A_j^{(1)}}\frac{\partial v_{i*}^{(1)}}{\partial\xi_j^{(1)}} - \eta^t Rk_j^{(1)}v_{j*}^{(1)}\right)$$
$$g_{i*}^{(1)} = \frac{1}{a_{i*}}\left(\eta^{e-t-d+c}\frac{1}{A_i^{(1)}}\frac{\partial v_{3*}^{(1)}}{\partial\xi_i^{(1)}} - \eta^c\frac{R}{R_i^{(1)}}v_{i*}^{(1)}\right)$$

$$L_{i*}^{(1)} = \frac{1}{A_i^{(1)}}\frac{\partial\sigma_{i*}^{(1)}}{\partial\xi_i^{(1)}} + \eta^{1-2t+c+a}\frac{1}{A_j^{(1)}}\frac{\partial\sigma_{ij*}^{(1)}}{\partial\xi_j^{(1)}} + \eta^t Rk_j^{(1)}(\sigma_{i*}^{(1)} - \sigma_{j*}^{(1)}) +$$
$$\eta^{1-t+c+a}Rk_i^{(1)}(\sigma_{ij*}^{(1)} + \sigma_{ji*}^{(1)})$$
$$L_*^{(1)} = \sigma_{1*}^{(1)}\frac{R}{R_1^{(1)}} + \sigma_{2*}^{(1)}\frac{R}{R_2^{(1)}}$$
$$F_*^{(1)} = \frac{1}{A_1^{(1)}}\frac{\partial\sigma_{13*}^{(1)}}{\partial\xi_1^{(1)}} + \frac{1}{A_2^{(1)}}\frac{\partial\sigma_{23*}^{(1)}}{\partial\xi_2^{(1)}} + \eta^t Rk_2^{(1)}\sigma_{13*}^{(1)} + \eta^t Rk_1^{(1)}\sigma_{23*}^{(1)}$$
$$a_{i*} = 1 + \eta^1\zeta\frac{R}{R_i}$$

Integrating Eqs.(2.7) in ζ, we obtain the following expansions:

$$\sigma_{i3*}^{(1)} = \sigma_{i3,0}^{(1)} + \eta^{d-c}\zeta\sigma_{i3,1}^{(1)} + \eta^{2d-2t}\zeta^2\sigma_{i3,2}^{(1)} + \dots \tag{2.9}$$
$$\sigma_{3*}^{(1)} = \sigma_{3,0}^{(1)} + \eta^{d-2t+c}\zeta\sigma_{3,1} + \eta^{2d-2t}\zeta^2\sigma_{3,2}^{(1)} + \dots \quad (\sigma_{i*}^{(1)}, p_*)$$
$$\sigma_{ij*}^{(1)} = \sigma_{ij,0}^{(1)} + \zeta\sigma_{ij,1}^{(1)} + \eta^{d-c}\zeta^2\sigma_{ij,2}^{(1)} + \dots \quad (v_{i*}^{(1)}, e_{i*}^{(1)}, m_{i*}^{(1)})$$
$$v_{3*}^{(1)} = v_{3,0}^{(1)} + \eta^{2d-2t-e}(\zeta v_{3,1}^{(1)} + \zeta^2 v_{3,2}^{(1)}) + \eta^{d-c}\zeta^3 v_{3,3}^{(1)} + \dots$$

Some of the above formulas are followed by expressions in parenthesis. These contain quantities whose expansions have exactly the same form.

We shall construct an approximate theory of two-layer shells, to within quantities of the order of ε

$$\varepsilon = O(\eta^{1-c}) \tag{2.10}$$

Substituting expansions (2.9) into (2.7) and (2.8) and neglecting terms of the order of smallness of up to (2.10) we obtain, after equating to zero the coefficients of like powers of ζ, the following relations:

$$\sigma_{i,0}^{(1)} = \eta^{d-2t+c}e_{i,0}^{(1)} + p_{,0}, \quad \sigma_{i,1}^{(1)} = \eta^{d-2t+c}e_{i,1}^{(1)} + p_{,1} \tag{2.11}$$
$$\eta^{d-2t+c}v_{3,1}^{(1)} = \sigma_{3,0}^{(1)} - p_{,0}, \quad \eta^{d-2t+c}2v_{3,2}^{(1)} = \sigma_{3,1}^{(1)} - p_{,1}$$
$$v_{i,1}^{(1)} = 2\sigma_{i3,0}^{(1)}, \quad v_{3,1}^{(1)} = -e_{i,0}^{(1)} - e_{j,0}^{(1)}$$
$$2v_{3,2}^{(1)} = -e_{1,1}^{(1)} - e_{2,1}^{(1)}, \quad \sigma_{i3}^{(1)} = \sigma_{i3,0}^{(1)}, \quad \sigma_{3,1}^{(1)} = -\eta^{d-2t+c}F_{,0}^{(1)}$$

The boundary conditions will be written with an accuracy of up to (2.10) thus:

$$\sigma_{i3,0}^{(1)} = q_{i3*}^{+}, \quad \sigma_{3,0}^{(1)} + \eta^{d-2t+c}\zeta\sigma_{3,1}^{(1)} = q_{3*}^{+} \tag{2.12}$$
$$v_{i,0}^{(1)} - \zeta_1 v_{i,1}^{(1)} = \eta^{a-d-1}v_{i*}^{(2)}, \quad v_{3,0}^{(1)} + \eta^{2d-2t-e}(-\zeta_1 v_{3,1}^{(1)} + \zeta_1^2 v_{3,2}^{(1)}) =$$
$$\eta^{a-e+c-3}v_{3*}^{(2)}, \quad \zeta_1 = -h_1/R$$

Returning from the asymptotic representations of the quantities required, introduced by formulas (2.3) and (2.4), to the dimensional stresses, strains and displacements, we obtain the following equation for the required quantities of the first layer (here, unlike in the quantities reduced to dimensionless form, we omit the superscript 1 within the parenthesis and the asterisks):

$$\sigma_{i3,0} = q_{i3}^{+}, \quad \sigma_{3,1} = -F_{,0} = -\left(\frac{1}{A_1}\frac{\partial q_{13}^{+}}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial q_{23}^{+}}{\partial\alpha_2} + k_1 q_{23}^{+} + k_2 q_{13}^{+}\right) \tag{2.13}$$
$$\sigma_{3,0} = q_3^{+} - h_1\sigma_{3,1}, \quad v_{i,1} = \frac{1}{\mu}\sigma_{i3,0}, \quad v_{i,0} = h_1 v_{i,1} + U_i$$
$$e_{i,0} = \frac{1}{A_i}\frac{\partial v_{i,0}}{\partial\alpha_i} + k_i v_{j,0} - \frac{v_{3,0}}{R_i}, \quad v_{3,1} = -e_{1,0} - e_{2,0}$$
$$e_{i,1} = \frac{1}{A_i}\frac{\partial v_{i,1}}{\partial\alpha_i} + k_i v_{j,1}, \quad v_{3,2} = -\frac{1}{2}(e_{1,1} + e_{2,1})$$
$$v_{3,0} = h_1 v_{3,1} - h_1^2 v_{3,2} - w, \quad p_{,0} = \sigma_{3,0} - 2\mu v_{3,1}$$
$$p_{,1} = \sigma_{3,1} - 4\mu v_{3,2}, \quad \sigma_{i,n} = \frac{1}{A_j}\frac{\partial v_{i,n}}{\partial\alpha_j} - k_j v_{j,n}$$
$$\sigma_{i,n} = 2\mu e_{i,n} + p_{,n}, \quad \sigma_{ij,n} = \mu(m_{i,n} + m_{j,n})$$
$$U_i = u_i - h_2\frac{1}{A_i}\frac{\partial w}{\partial\alpha_i}, \quad i \neq j = 1, 2; \quad n = 0, 1$$

where u_i, w are the components of displacement vector of the middle surface of metal layer. Formulas (2.13) determine completely all the quantities required in terms of the surface load applied to the face surface of the elastomer layer, and in terms of the displacement of the contact surface between the layers. From formulas (2.13) we see that the following stresses act in the contact surface:

$$\sigma_{i3}|_\Gamma = q_{i3}^{+}, \quad \sigma_3|_\Gamma = q_3^{+} + 2h_1\left(\frac{1}{A_1}\frac{\partial q_{13}^{+}}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial q_{23}^{+}}{\partial\alpha_2} + k_1 q_{23}^{+} + k_2 q_{13}^{+}\right) \tag{2.14}$$

This means that the tangential component of the load is transmitted through the layer of elastomer to the metal layer without any change, while the normal component changes the more, the larger the variability of the load over the coordinates. If the variability coefficient is equal to zero, we can assume, within the approximation adopted here, that the normal load at the contact surface is the same as that at the surface of the elastomer $\sigma_3|_\Gamma = q_3^+$. When the variability coefficient t increases, the contribution of the term which takes into account the correction to the surface load (the term within the parenthesis in (2.14)) also increases and becomes, when $t = 1/2$, commensurate with the principal term q_3^+. We note that for the stress states with large variability, the value of the normal stresses at the contact surface depends essentially on the tangential load acting on the face surface of the soft layer.

Thus the static computation of a two-layer, resin-metal shell, can be carried out in two stages. The first stage involves computing a single layer metal shell using classical theory, with the following surface load:

$$X_i = (q_{i3}^+ + q_{i3}^-)$$

$$Z = -(q_3^+ + q_3^-) - 2h_1\left(\frac{1}{A_1}\frac{\partial q_{13}^+}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial q_{23}^+}{\partial \alpha_2} + k_1 q_{23}^+ + k_2 q_{13}^+\right)$$

After solving the first problem, we compute the elastomer layer using direct actions according to the formulas (2.13), in the order in which the formulas have been written.

The results obtained can be generalized to the case of a slightly compressible elastomer. To do this we must introduce an asymptotic estimate of the compressibility of the material $(1-2\nu) = \eta^q$ (ν is Poisson's ratio), in which case we must replace the equation of incompressibility by

$$e_{1*}^{(1)} + e_{2*}^{(1)} + \frac{\partial v_{3*}^{(1)}}{\partial \zeta} = \eta^{q-1+2t+c} p_*^{(1)}$$

The last formula shows that all previous formulas remain valid for the power indices $s = q - 1 + 2t + c > 0$ with an accuracy of up to the order of $O(\eta^s + \eta^{1-c})$. If $s = 0$, then asymptotic analysis shows that the formulas for determining $v_{3,1}$, $v_{3,2}$ in (2.13) are replaced by

$$v_{3,1} = \frac{1-2\nu}{1+\nu}\frac{1}{2\mu}p_{,0} - e_{1,0} - e_{2,0}, \qquad 2v_{3,2} = \frac{1-2\nu}{1+\nu}\frac{1}{2\mu}p_{,1} - e_{1,1} - e_{2,1}$$

and the order in which the quantities required are determined using algebraic actions, will also change. The asymptotic expression (2.3) can be used only for non-negative values of s. We shall confine ourselves to the compressibility which satisfies the above requirement, and this will correspond to the properties of real elastomers.

3. Let us assume that the surface load (1.6), (1.7) varies sinusoidally as $e^{-i\omega\tau}$ where τ is the time and ω is the angular frequency of the oscillation. We introduce a dimensionless frequency parameter for the stiff layer

$$\rho_2\omega^2 R^2/E = \eta^{2r}\lambda \tag{3.1}$$

Let us write the dimensionless frequency parameter for the soft layer in terms of λ, taking (2.1) into account

$$\rho_1\omega^2 R^2/(2\mu) = \eta^{2r-a+b}\lambda \tag{3.2}$$

Substituting the asymptotic expressions (3.1) and (3.2) into the equations of motion we can show that when $z = 2r + 2 - a + b > 0$, we have, with an accuracy of order $O(\eta^{1-c} + \eta^z)$, a quasistatic problem and the approximate formulas obtained for the static problem will remain valid. If $z \leqslant 0$, then we have a dynamic problem, and in this connection we must take into account the inertial terms in the equations of equilibrium. Moreover, when the z are negative, dynamic integrals of the quantities required will appear, in which the variability of y over the thickness coordinate will be greater than unity:

$$y = (a - b - 2r)/2$$

In this case we shall have the asymptotic representation (2.2)-(2.4) and three-dimensional equations for the soft layer will be written, taking the asymptotic representations into account, in the form (2.7), (2.8).

Let us write, with an accuracy to quantities of the order of $O(\eta^{y-c})$, the approximate equations for determining the unknown quantities of the soft layer

$$\frac{\partial \sigma_{i3}}{\partial \alpha_3} + \rho_1\omega^2 v_i = 0, \qquad \sigma_{i3} = \mu\frac{\partial v_i}{\partial \alpha_3} \tag{3.3}$$

$$e_i = \frac{1}{A_i}\frac{\partial v_i}{\partial \alpha_i} + k_i v_j, \qquad \frac{\partial v_3}{\partial \alpha_3} = -e_1 - e_2$$

$$F_{,0} = \frac{1}{A_1}\frac{\partial \sigma_{13}}{\partial \alpha_1} + \frac{1}{A_2}\frac{\partial \sigma_{23}}{\partial \alpha_2} + k_2\sigma_{13} + k_1\sigma_{23}$$

$$\frac{\partial \sigma_3}{\partial \alpha_3} = -F_{,0} - \rho_1 \omega^2 v_3, \quad p = \sigma_3 - 2\mu \frac{\partial v_3}{\partial \alpha_3}$$

$$\sigma_i = 2\mu e_i + p, \quad m_i = \frac{1}{A_j} \frac{\partial v_i}{\partial \alpha_j} - k_j v_j, \quad \sigma_{ij} = \mu (m_i + m_j)$$

Eqs.(3.3) can be integrated in general form, and the arbitrary constants of integration can be found from the conditions on the face and contact surfaces. The resulting differential equation of the system (3.3) will have the form

$$\frac{\partial^2 v_i}{\partial \alpha_3^2} + \frac{\rho_1 \omega^2}{\mu} v_i = 0$$

and from this we have

$$v_i = C_1 \sin k\alpha_3 + C_2 \cos k\alpha_3, \; \sigma_{i3} = \mu k (C_1 \cos k\alpha_3 - C_2 \sin k\alpha_3)$$

$$k = \sqrt{\rho_1 \omega^2 / \mu}$$

From the condition $v_i |_{\Gamma_1} = U_i, \; \sigma_{i3} |_{\Gamma_1} = q_{i3}^+$ we obtain

$$2C_1 = (U_i \sin kh_1 - (\mu k)^{-1} q_{i3}^+ \cos kh_1)/\cos 2kh_1, \quad 2C_2 = (U_i \cos kh_1 + (\mu k)^{-1} q_{i3}^+ \sin kh_1)/\cos 2kh_1$$

Note that in the case of a linearly elastic elastomer we have thickness resonances with frequency

$$\omega_r = {}^1/_2 \pi h_1^{-1} \sqrt{\mu \rho_1^{-1}} \, ({}^1/_2 + n), \; n = 0, \pm 1, \pm 2, \ldots$$

The remaining unknown quantities can be found from formulas (3.3) taking conditions (1.6), (1.8) into account.

The solution of the rigid layer reduces, just as in the static case, to calculating a single-layer shell acted upon by the load

$$X_i = (\sigma_{i3}^{(1)} |_\Gamma + q_{i3}^-), \; Z = (\sigma_3^{(1)} |_\Gamma + q_3^-)$$

where σ_{i3}, σ_3 are expressed, as a result of solving the problem formulated above, in terms of the displacement of the middle surface of the rigid layer and surface load q_{i3}^+, q_3^+.

In order to take into account the limited compressibility, it is sufficient to change one of the Eqs.(3.3):

$$\frac{\partial v_3}{\partial \alpha_3} = -e_1 - e_2 + \frac{1-2\nu}{1+\nu} \frac{1}{2\mu} p$$

REFERENCES

1. CHERNYKH K.F. and MILYAKOVA L.V., Thin resin-metal elements. Vestn. LGU, 19, 4, 1981.
2. LAVENDEL E.E., KHRICHIKOVA V.A. and LEIKAND M.A., Computing the compressive strength of thin-layer resin-metal elements. Voprosy Dinamiki i Prochnosti. Zinatne, Riga,38, 1981.
3. MAL'KOV V.M., Linear theory of a thin layer of a slightly compressible material. Dokl. Akad. Nauk SSSR, 293, 1, 1987.
4. GOL'DENVEIZER A.L., Theory of Thin Elastic Shells. Nauka, Moscow, 1976.

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